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Twisted Yangians and folded \mathcal{W} -algebras

E. Ragoucy*

Laboratoire de Physique Théorique LAPTH[†]

LAPP, BP 110

F-74941 Annecy-le-Vieux Cedex, France.

Abstract

We show that the truncation of twisted Yangians are isomorphic to finite \mathcal{W} -algebras based on orthogonal or symplectic algebras. This isomorphism allows us to classify all the finite dimensional irreducible representations of the quoted \mathcal{W} -algebras. We also give an R-matrix for these \mathcal{W} -algebras, and determine their center.

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*ragoucy@lapp.in2p3.fr

[†]UMR 5108 du CNRS, associée à l'Université de Savoie.

1 Introduction

Recently [1, 2], it has been remarked that Yangians [3] and \mathcal{W} -algebras [4] (both based on $gl(N)$ algebras) can be connected, although they belong to different fields of theoretical physics.

\mathcal{W} -algebras have been introduced in the $2d$ -conformal models as a tool for the study of these theories, such as the Toda field theories [5]. Then, these algebras and their finite-dimensional versions appeared to be relevant in several physical backgrounds [6, 7]. However, a full understanding of their algebraic structure (and of their geometrical interpretation) is lacking.

Yangians were first considered and defined in connection with some rational solutions of the quantum Yang-Baxter equation. Later, their relevance in integrable models with non Abelian symmetry was remarked [8].

The connection of some of the finite \mathcal{W} -algebras with Yangians (both based on $gl(N)$ algebras) appears to shed some light on the algebraic structure of the former: it allows the construction of an R -matrix for \mathcal{W} -algebras, the classification of their irreducible finite-dimensional representations and the determination of their center.

In the present article, we continue with the study of this connection for the case of finite \mathcal{W} -algebras based on orthogonal and symplectic algebras. Such \mathcal{W} -algebras appear to be truncations of twisted Yangians [9, 10]. Using this relation, we give an R -matrix formulation for the corresponding \mathcal{W} -algebras. Contrarily to the RTT formulation encountered in the case of $gl(N)$, it appears here to be of $ABCD$ -algebras type [11], *i.e.* an $RSR'S$ formulation. Since these algebras are not Hopf algebras, this confirms the remark that \mathcal{W} -algebras seem to have *no* natural Hopf structure. The $RSR'S$ formulation allows to classify the irreducible finite-dimensional representations of the \mathcal{W} -algebras and to determine their center.

The article is organized as follows: in section 2, we remind some notions on Yangians and finite \mathcal{W} -algebras. In section 3, we present the construction that lead to twisted Yangians, as it was originally done in [10], and summarize some of their properties, such as the classification of finite-dimensional irreducible representation. In section 4, we apply the same type of procedure to finite \mathcal{W} -algebras, to obtain what is known as folded \mathcal{W} -algebras [12]. The comparison between these two objects is done in section 5, and is used to classify the finite-dimensional representations of \mathcal{W} -algebras. We conclude in section 6.

2 Yangians and \mathcal{W} -algebras based on $gl(Np)$

We briefly present some known results on Yangians [3] and finite \mathcal{W} -algebras [7] that will be used in the following.

2.1 The Yangian $Y(gl(N)) \equiv Y(N)$

One starts with the Yangian $Y(N)$ based on $gl(N)$. It is a Hopf algebra, the structure of which is contained in the relations (see for instance [13] and ref. therein for more details):

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v) \quad (2.1)$$

$$\Delta(T(u)) = T(u) \otimes T(u) \quad (2.2)$$

We use here the usual convention on auxiliary spaces $T_1 = T \otimes \mathbb{I}$ and $T_2 = \mathbb{I} \otimes T$. The generators $T_{(n)}^{ij}$ of the Yangian are gathered in

$$T(u) = \sum_{n=0}^{\infty} \sum_{i,j=1}^N u^{-n} T_{(n)}^{ij} E_{ij} = \sum_{n=0}^{\infty} u^{-n} T_{(n)} = \sum_{i,j=1}^N T^{ij}(u) E_{ij} \text{ with } T_{(0)}^{ij} = \delta^{ij} ; T_{(0)} = \mathbb{I} \quad (2.3)$$

where E_{ij} is the $N \times N$ matrix with 1 at position (i, j) , and $\mathbb{I} = \sum_{i=1}^N E_{ii}$.

The R -matrix is given by

$$R_{12}(x) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} P_{12} ; P_{12} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \quad (2.4)$$

P_{12} is the permutation of the two auxiliary spaces.

The relation (2.1) is equivalent to the commutation relations:

$$[T_{(m)}^{ij}, T_{(n)}^{kl}] = \sum_{r=0}^{\min(m,n)-1} \left(T_{(r)}^{kj} T_{(m+n-r-1)}^{il} - T_{(r)}^{il} T_{(m+n-r-1)}^{kj} \right) \quad (2.5)$$

or keeping the auxiliary spaces:

$$[T_{(m)1}, T_{(n)2}] = \sum_{r=0}^{\min(m,n)-1} \left(P_{12} T_{(r)1} T_{(m+n-r-1)2} - T_{(r)2} T_{(m+n-r-1)1} P_{12} \right) \quad (2.6)$$

(n) defines a natural gradation on the Yangian, and we will call level the corresponding grade (*i.e.* $T_{(n)}^{ij}$ and $T_{(n)}$ are said of level n).

The center $Z(N)$ of $Y(N)$ has been determined in [10, 9]. It is generated by the quantum determinant:

$$\text{qdet}[T(u)] = \prod_{\sigma \in S_N} (-1)^{\text{sg}(\sigma)} T_{1\sigma(1)}(u) T_{2\sigma(2)}(u-1) \dots T_{N\sigma(N)}(u-N+1) = 1 + \sum_{n>0} d_n u^{-n} \quad (2.7)$$

Denoting by $SY(N)$ the Hopf algebra $Y(N)/Z(N)$, we have:

$$Y(N) \equiv Z(N) \otimes SY(N) \text{ and } SY(N) \equiv Y(sl(N)) \quad (2.8)$$

where $Y(sl(N))$ is the Yangian based on $sl(N)$.

The finite dimensional representations of $Y(N)$ has also been determined in [14, 15], see also [16, 17] for more details.

2.1.1 Classical version

One can take the classical limit of the Yangian:

$$R(x) = \mathbb{I} \otimes \mathbb{I} - \hbar r(x) ; T(u) = L(u) ; [\cdot, \cdot] = \hbar \{\cdot, \cdot\} \quad (2.9)$$

to get a Poisson Bracket

$$\{L_1(u), L_2(v)\} = [r_{12}(u-v), L_1(u)L_2(v)] \quad (2.10)$$

This procedure defines a classical (Poisson bracket) version of the Yangian, with an Abelian product $L_{(m)}^{ij} L_{(n)}^{kl} = L_{(n)}^{kl} L_{(m)}^{ij}$.

2.2 The $\mathcal{W}(gl(Np), N.sl(p)) \equiv \mathcal{W}_p(N)$ algebra

It is defined as an Hamiltonian reduction of $gl(Np)$, considered as a Poisson algebra (*i.e.* with Poisson brackets). A basis of $gl(Np)$ (see [2] for more details) consists in generators J_{jm}^{ab} , with $-j \leq m \leq j$, $0 \leq j \leq p$, and $a, b = 1, \dots, N$, submitted to

$$\{J_{jm}^{ab}, J_{\ell n}^{cd}\} = \sum_{r=|j-\ell|}^{j+\ell} \sum_{s=-r}^r \left(\delta^{bc} < j, m; \ell, n | r, s > J_{r,s}^{ad} - \delta^{ad} < \ell, n; j, m | r, s > J_{r,s}^{cb} \right) \quad (2.11)$$

$< j, m; \ell, n | r, s >$ are some Clebsch-Gordan like coefficients, defined by

$$< j, m; \ell, n | r, s > = \frac{(-1)^s}{N \eta_r} \text{tr} (M_{j,m}^{ab} \cdot M_{\ell,n}^{bc} \cdot M_{r,-s}^{ca}) \quad \text{with } \eta_r = (2r)!(r!)^2 \binom{p+r}{2r+1}$$

where M_{jm}^{ab} are $(Np) \times (Np)$ matrices representing J_{jm}^{ab} in the fundamental of $gl(Np)$. They have being given in [2], and we take the same normalizations:

$$\begin{aligned} M_{jm}^{ab} &= E^{ab} \otimes \left(\sum_{k=1}^{p-m} a_{j,m}^k E_{k,k+m} \right) ; a_{j,m}^k = \sum_{i=0}^{j-m} (-1)^{i+j+m} \binom{j-m}{i} a_{j,j}^{k-i} \text{ for } m \geq 0 \\ M_{jm}^{ab} &= E^{ab} \otimes \left(\sum_{k=1}^{p+m} a_{j,m}^k E_{k-m,k} \right) ; a_{j,m}^k = \sum_{i=0}^{j-m} (-1)^{i+j+m} \binom{j-m}{i} a_{j,j}^{k-i-m} \text{ for } m \leq 0 \\ a_{j,j}^k &= \frac{(k+j-1)!(p-k)!}{(k-1)!(p-k-j)!} \end{aligned}$$

where E^{ab} are $N \times N$ matrices and $E_{k\ell}$ are $p \times p$ ones. Note that we have the properties [2]:

$$a_{j,-j}^k = a_{j,-j}^0 = (-1)^j (2j)! \quad (2.12)$$

$$a_{j,m}^k = (-1)^{j+m} a_{j,m}^{N+1-k-m} \quad (2.13)$$

$$a_{j,m}^k = 0 \text{ for } |m| > j \quad (2.14)$$

On $gl(Np)$ we impose a set of second class constraints

$$\begin{aligned} J_{jm}^{ab} &= 0 \text{ for } m < 0, \forall j, a, b \text{ but } m = -1, j = 1 \\ J_{1,-1}^{ab} &= \delta^{ab} \forall a, b \end{aligned} \quad (2.15)$$

which will be denoted by $\Phi = \{\varphi_\alpha\}_{\alpha \in I}$ for convenience.

The \mathcal{W} -algebra is defined as the enveloping algebra of the generators $J_{jj}^{ab} \equiv W_j^{ab}$ equipped with the Dirac brackets associated to the constraints (2.15):

$$\{X, Y\}_* \sim \{X, Y\} - \sum_{\alpha, \beta \in I} \{X, \varphi_\alpha\} \mathcal{C}^{\alpha\beta} \{\varphi_\beta, Y\} \quad \forall X, Y \quad (2.16)$$

where the matrix $\mathcal{C}^{\alpha\beta}$ is the inverse of the matrix of constraints:

$$\mathcal{C}_{\alpha\beta} \sim \{\varphi_\alpha, \varphi_\beta\} ; \mathcal{C}^{\alpha\beta} \mathcal{C}_{\beta\gamma} \sim \mathcal{C}_{\alpha\gamma} \quad (2.17)$$

The symbol \sim means that one has to apply the constraints on the right hand side of each expression *once the Poisson Brackets have been computed*.

2.2.1 The $\mathcal{W}_p(N)$ algebra in the Yangian basis

In [2, 1], it has been shown that the $\mathcal{W}(gl(Np), N.sl(p)) \equiv \mathcal{W}_p(N)$ algebras are truncations of the Yangian $Y(N)$. They are thus defined by the following relations*:

$$T(u) = \sum_{n=0}^p \sum_{i,j=1}^N u^{-n} T_{(n)}^{ij} \otimes E_{ij} ; T_{(0)}^{ij} = \delta^{ij} \quad (2.18)$$

$$T_1(u) = T(u) \otimes \mathbb{I} ; T_2(u) = \mathbb{I} \otimes T(u) ; r(x) = \frac{1}{x} P_{12} \quad (2.19)$$

$$\{T_1(u), T_2(v)\} = [r_{12}(u-v), T_1(u)T_2(v)] \quad (2.20)$$

or equivalently

$$\{T_{(m)}^{ij}, T_{(n)}^{kl}\} = \sum_{r=0}^{\min(m,n,p)-1} \left(T_{(r)}^{kj} T_{(m+n-r)}^{il} - T_{(r)}^{il} T_{(m+n-r)}^{kj} \right) \quad (2.21)$$

There quantization becomes very simple in this basis. It reads:

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v) \quad (2.22)$$

$$R_{12}(x) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} P_{12} \quad (2.23)$$

with the same definition (2.18) of $T(u)$.

This connection with the Yangian $Y(N)$ allows to classify all the finite dimensional irreducible representations of $\mathcal{W}_p(N)$ and to determine its center [2].

*Note that, due to conflicting notations between \mathcal{W} -algebras and Yangians, W_n^{ij} corresponds to $T_{(n+1)}^{ij}$.

3 Twisted Yangians

3.1 Presentation of $Y^\pm(N)$

As the $gl(N)$ algebra, $Y(N)$ possesses automorphisms, and in the same way one can reconstruct the $so(n)$ and $sp(2n)$ algebras from the $gl(N)$ ones, one is tempted to reconstruct the Yangians based on orthogonal and symplectic algebras from $Y(N)$. However, the situation appears to be more delicate in the case of Yangians. In fact, although an algebraic structure similar to the one of orthogonal and symplectic Yangians can be achieved using automorphisms, the resulting algebra is not isomorphic to these Yangians. It is even not a Hopf algebra. The construction has been defined in [9, 10].

More precisely, the automorphisms we consider on $Y(N)$ are of the form

$$\begin{aligned} \tau(T(u)) &= T^t(-u) \text{ with } T^t(u) = \sum_{i,j=1}^N T^{ij}(u) E_{ij}^t \text{ and } E_{ij}^t = \theta^i \theta^j E_{N+1-j, N+1-i} \\ \tau(T_{(m)}^{ij}) &= (-1)^m \theta^{N+1-i} \theta^{N+1-j} T_{(m)}^{N+1-j, N+1-i} ; \quad \theta^i = \pm 1 \end{aligned} \quad (3.1)$$

Asking the automorphism τ to be of order 2 leads to the constraint

$$\theta^i \theta^{N+1-i} = \theta_0 \text{ with } \theta_0 = \pm 1 \quad (3.2)$$

If $N = 2n + 1$, equation (3.2) for $i = n + 1$ implies $\theta_0 = 1$. We thus have the following conditions on θ_0 :

$$\theta_0 = 1 \text{ for } N = 2n + 1 \text{ and } \theta_0 = \pm 1 \text{ for } N = 2n \quad (3.3)$$

Each allowed values of the parameters θ^i determine an automorphism τ . However, only the values of θ_0 are relevant for our purpose, as we will see in the following.

Once τ is chosen, one defines the following generators in $Y(N)$

$$S(u) = T(u) \tau(T(u)) \quad (3.4)$$

From the relations (2.1), one deduces (see [10] for more details)

$$R_{12}(u - v) S_1(u) R'_{12}(u + v) S_2(v) = S_2(v) R'_{12}(u + v) S_1(u) R_{12}(u - v) \quad (3.5)$$

where

$$\begin{aligned} R'(x) &= (\tau \otimes id)(R(x)) = (id \otimes \tau)(R(x)) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} Q_{12} \\ \text{with } Q_{12} &= \sum_{i,j=1}^N \theta^i \theta^j E_{ij} \otimes E_{N+1-i, N+1-j} = (\tau \otimes id)(P_{12}) \end{aligned} \quad (3.6)$$

Note that this construction is a particular case of the $abcd$ -algebras introduced in [11], and was indeed mentioned there as an example. The full structure of twisted Yangians has been studied in [10].

The above relation defines a subalgebra $Y^\tau(N)$ of the Yangian $Y(N)$. Looking at the generators of $Y^\tau(N)$

$$S(u) = \sum_{i,j=1}^N u^{-n} S_{(n)}^{ij} E_{ij} = \sum_{n=0}^{\infty} u^{-n} S_{(n)} = \sum_{i,j=1}^N S^{ij}(u) E_{ij} \text{ with } S_{(0)}^{ij} = \delta^{ij} \quad (3.7)$$

one can show that they obey the following relations:

$$\begin{aligned} [S_{(m)}^{ij}, S_{(n)}^{kl}] &= \sum_{r=0}^{\min(m,n)-1} \left[S_{(r)}^{kj} S_{(m+n-r-1)}^{il} - S_{(m+n-r-1)}^{kj} S_{(r)}^{il} + \right. \\ &\quad \left. + (-)^{n+r} \theta_0 \left(\theta^k \theta^j S_{(r)}^{i,N+1-k} S_{(m+n-r-1)}^{N+1-j,l} - \theta^i \theta^l S_{(m+n-r-1)}^{k,N+1-i} S_{(r)}^{N+1-l,j} \right) \right] + \\ &\quad + \sum_{s=0}^{\lfloor \frac{m-2}{2} \rfloor} \theta_0 \theta^j \theta^i \left(S_{(m-2s-2)}^{k,N+1-i} S_{(n+2s)}^{N+1-j,l} - S_{(n+2s)}^{k,N+1-i} S_{(m-2s-2)}^{N+1-j,l} \right) \end{aligned} \quad (3.8)$$

which is equivalent to (3.5), as well as

$$\begin{aligned} [S^{ij}(u), S^{kl}(v)] &= \frac{1}{u-v} \left(S^{kj}(u) S^{il}(v) - S^{kj}(v) S^{il}(u) \right) + \\ &\quad - \frac{\theta_0}{u+v} \left(\theta^k \theta^j S^{i,N+1-k}(u) S^{N+1-j,l}(v) - \theta^i \theta^l S^{k,N+1-i}(v) S^{N+1-l,j}(u) \right) + \\ &\quad + \frac{\theta_0 \theta^j \theta^i}{u^2 - v^2} \left(S^{k,N+1-i}(u) S^{N+1-j,l}(v) - S^{k,N+1-i}(v) S^{N+1-j,l}(u) \right) \end{aligned} \quad (3.9)$$

and also

$$\begin{aligned} [S_1(u), S_2(v)] &= \frac{1}{u-v} \left(P_{12} S_1(u) S_2(v) - S_2(v) S_1(u) P_{12} \right) + \\ &\quad - \frac{1}{u+v} \left(S_1(u) Q_{12} S_2(v) - S_2(v) Q_{12} S_1(u) \right) + \\ &\quad + \frac{1}{u^2 - v^2} \left(P_{12} S_1(u) Q_{12} S_2(v) - S_2(v) Q_{12} S_1(u) P_{12} \right) \end{aligned}$$

Theorem 1 *All the θ^i dependence, but θ_0 , can be removed. We thus have only one (two) different twisted Yangian(s), corresponding to the (two) value(s) $\theta_0 = 1$ ($\theta_0 = \pm 1$) for $N = 2n + 1$ (for $N = 2n$).*

Proof: We prove this by exhibiting a basis in which the θ^i dependence, but θ_0 , has disappeared. In fact, the twisted Yangians are generated by

$$\begin{aligned} Y^\pm(2n) & \quad S^{ij}(u) ; S^{i,N+1-j}(u) ; S^{N+1-i,j}(u) \quad i, j = 1, \dots, n \\ Y^+(2n+1) & \quad \begin{cases} S^{ij}(u) ; S^{i,N+1-j}(u) ; S^{N+1-i,j}(u) \\ S^{n+1,n+1}(u) ; S^{n+1,i}(u) ; S^{i,n+1}(u) ; i, j = 1, \dots, n \end{cases} \end{aligned} \quad (3.10)$$

One then defines for $Y^\pm(2n)$:

$$J^{ij}(u) = \theta^i \theta^j S^{ij}(u) ; K^{ij}(u) = \theta^i S^{i, N+1-j}(u) ; \bar{K}^{ij}(u) = \theta^j S^{N+1-i, j}(u) \quad (3.11)$$

which obey to θ -free commutation relations (except the θ_0 dependence). For instance:

$$[J^{ij}(u), J^{kl}(v)] = \frac{1}{u-v} \left(J^{kj}(u) J^{il}(v) - J^{kj}(v) J^{il}(u) \right) + \quad (3.12)$$

$$- \frac{\theta_0}{u+v} \left(K^{i,k}(u) \bar{K}^{j,l}(v) - K^{k,i}(v) \bar{K}^{l,j}(u) \right) + \quad (3.13)$$

$$+ \frac{\theta_0}{u^2 - v^2} \left(K^{k,i}(u) \bar{K}^{j,l}(v) - K^{k,i}(v) \bar{K}^{j,l}(u) \right) \quad (3.14)$$

For $Y^+(2n+1)$, the redefinition is the same as before, plus for the remaining generators:

$$J_0(u) = \theta^{n+1} S^{n+1, n+1}(u) ; L^i(u) = \theta^i S^{n+1, i}(u) ; \bar{L}^i(u) = \theta^{n+1} \theta^i S^{i, n+1}(u) \quad (3.15)$$

The commutation relations are then free from any θ 's. \blacksquare

Definition 1 *The twisted Yangians $Y^\pm(N)$ correspond to the following choices for τ :*

$$\begin{aligned} \text{For } Y^+(N) : \quad & \theta^i = 1, \forall i \quad \text{i.e. } \theta_0 = 1 \\ \text{For } Y^-(2n) : \quad & \theta^i = \text{sg}\left(\frac{N+1}{2} - i\right), \forall i \quad \text{i.e. } \theta_0 = -1 \end{aligned} \quad (3.16)$$

With these choices, the Lie subalgebra for $Y^\pm(N)$ is

$$\mathfrak{so}(N) \subset Y^+(N) \quad \text{and} \quad \mathfrak{sp}(2n) \subset Y^-(2n) \quad (3.17)$$

Denoting by $Y_k^\pm(N)$ the subset of $Y^\pm(N)$ formed by the generators of level k , we have

$$\dim Y_{2k-1}^\pm(N) = \frac{N(N \mp 1)}{2} ; \dim Y_{2k}^\pm(N) = \frac{N(N \pm 1)}{2}, \quad k = 1, 2, \dots \quad (3.18)$$

In [10], it has been proven:

Property 1 *The $Y(N)$ -subalgebra $Y^\tau(N)$ generated by $S(u) = T(u)\tau(T(u))$ is isomorphic to the algebra defined by the two following relations:*

$$R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v) \quad (3.19)$$

$$\tau[S(u)] = S(u) + \theta_0 \frac{S(u) - S(-u)}{2u} \quad (3.20)$$

Let us remark that one can perform the change:

$$S'(u) = S(u) + \theta_0 \frac{S(u) - S(-u)}{4u} = \frac{1}{2} \left(S(u) + \tau[S(u)] \right) \quad (3.21)$$

which obeys $\tau[S'(u)] = S'(u)$. This proves that twisted Yangians are a subalgebra of $\text{Ker}(id - \tau)$, the subalgebra generated by elements of $Y(N)$ which are invariant under τ . However, the commutation relations satisfied by $S'(u)$ are more complicated than the ones obeyed by $S(u)$. We will come back on this point in the classical case (see below).

3.2 Center of $Y^\pm(N)$

The center of the twisted Yangian has been studied in [9, 10]. It is a true subalgebra of the center of $Y(N)$, and is generated by the so-called Sklyanin determinant sdet . The exact expression of the Sklyanin determinant in term of $S(u)$ can be found in [18]. It is however a rather complicated expression. A more easy-to-handle expression, which refers to the underlying $Y(N)$, can be found in [9, 10]:

$$\begin{aligned} \text{sdet}[S(u)] &= \gamma_N(u) \text{qdet}[T(u)] \text{qdet}[T(N-1-u)] \quad \text{with :} \\ \gamma_N(u) &= 1 \quad \text{for } Y^+(N) \quad \text{and} \quad \gamma_N(u) = \frac{2u+1}{2u+1-N} \quad \text{for } Y^-(N) \end{aligned} \quad (3.22)$$

One can show [18, 10] that the Sklyanin determinant provides a basis c_2, c_4, \dots for the center of $Y^\pm(N)$ through the formulae:

$$\begin{aligned} \text{sdet}\left[T\left(u - \frac{N+1}{2}\right)\right] &= 1 + \sum_{n>0} c_{2n} u^{-2n} \quad \text{for } Y^+(N) \\ \text{sdet}\left[T\left(u + n - \frac{1}{2}\right)\right] &= (1 + n u^{-1}) \left(1 + \sum_{n>0} c_{2n} u^{-2n}\right) \quad \text{for } Y^-(2n) \end{aligned} \quad (3.23)$$

If one denotes by $Z^\pm(N)$ the center of $Y^\pm(N)$, one has $Y^\pm(N) \equiv Z^\pm(N) \otimes SY^\pm(N)$, where the special twisted Yangian $SY^\pm(N)$ is defined by $SY^\pm(N) = SY(N) \cap Y^\pm(N)$.

3.3 Classical case

As for the Yangian, one can take a classical limit of the twisted Yangian

$$\{S_1(u), S_2(v)\} = [r_{12}(u-v), S_1(u)S_2(v)] + S_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)S_2(v) \quad (3.24)$$

where $r'_{12}(x) = (id \otimes \tau)(r(x)) = (\tau \otimes id)(r(x)) = \frac{1}{x}Q_{12}$, or more explicitly

$$\begin{aligned} \{S_1(u), S_2(v)\} &= \frac{1}{u-v} \left(P_{12} S_1(u) S_2(v) - S_2(v) S_1(u) P_{12} \right) + \\ &\quad - \frac{1}{u+v} \left(S_1(u) Q_{12} S_2(v) - S_2(v) Q_{12} S_1(u) \right) \end{aligned}$$

It leads to the following Poisson brackets for the generators:

$$\begin{aligned} \{S_{(m)}^{ij}, S_{(n)}^{kl}\} &= \sum_{r=0}^{\min(m,n)-1} \left[S_{(r)}^{kj} S_{(m+n-r-1)}^{il} - S_{(m+n-r-1)}^{kj} S_{(r)}^{il} + \right. \\ &\quad \left. + (-)^{n+r} \theta_0 \left(\theta^k \theta^j S_{(r)}^{i, N+1-k} S_{(m+n-r-1)}^{N+1-j, l} - \theta^i \theta^l S_{(m+n-r-1)}^{k, N+1-i} S_{(r)}^{N+1-l, j} \right) \right] \end{aligned} \quad (3.25)$$

Note that (3.24) can be defined as the classical limit of the twisted Yangian as well as the twisted subalgebra of the classical Yangian, with still $S(u) = L(u)\tau(L(u))$.

Of course, as in the quantum case, it is only the θ_0 dependence which is relevant, and we will deal with classical twisted Yangian $Y^\pm(N)$ only.

Property 2 *The classical $Y^\tau(N)$ algebra, generated by $S(u) = T(u)\tau(T(u))$, is isomorphic to the $Y(N)$ -subalgebra defined by the two following relations:*

$$\begin{aligned} \{S_1(u), S_2(v)\} &= [r_{12}(u-v), S_1(u)S_2(v)] + S_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)S_2(v) \\ \tau(S(u)) &= S(u) \end{aligned}$$

The classical twisted Yangians $Y^\tau(N)$ are thus isomorphic to subalgebras of the τ -invariant subalgebra $\text{Ker}(id - \tau)$ in $Y(N)$.

Proof: We start with the classical version of the property 1 (proven in [10]): $Y^\pm(N)$ is completely defined by the relation (3.24), together with

$$\tau[S(u)] = S(u) + \theta_0 \frac{S(u) - S(-u)}{2u} \quad (3.26)$$

Now, defining

$$S'(u) = S(u) \pm \frac{S(u) - S(-u)}{4u} = \frac{1}{2} (S(u) + \tau(S(u))) \quad (3.27)$$

one computes that $S'(u)$ still obey the quadratic relation (3.24) with now as symmetry relation $\tau(S'(u)) = S'(u)$. Indeed, starting from the quadratic relation on $S(u)$, and denoting $\bar{S} = \tau(S)$, one deduces:

$$\begin{aligned} \{\bar{S}_1(u), S_2(v)\} &= [r_{12}(u-v), \bar{S}_1(u)S_2(v)] + S_2(v)r'_{12}(u+v)\bar{S}_1(u) - \bar{S}_1(u)r'_{12}(u+v)S_2(v) \\ \{S_1(u), \bar{S}_2(v)\} &= [r_{12}(u-v), S_1(u)\bar{S}_2(v)] + \bar{S}_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)\bar{S}_2(v) \\ \{\bar{S}_1(u), \bar{S}_2(v)\} &= [r_{12}(u-v), \bar{S}_1(u)\bar{S}_2(v)] + \bar{S}_2(v)r'_{12}(u+v)\bar{S}_1(u) - \bar{S}_1(u)r'_{12}(u+v)\bar{S}_2(v) \end{aligned}$$

where one has heavily used the commutativity of the product at the classical level. From these formulae, it is simple to deduce that $S' = \frac{1}{2}(S + \bar{S})$ also obeys (3.24).

Finally, the calculation

$$S'(u) - S'(-u) = S(u) - S(-u) \quad (3.28)$$

shows that the change from $S(u)$ to $S'(u)$ is invertible. ■

In the basis presented in theorem 1, the condition $\tau[S(u)] = S(u)$ reduces (at classical level) to $K^{ij}(u) = \theta_0 K^{ji}(-u)$, $\bar{K}^{ij} = \theta_0 \bar{K}^{ji}(-u)$, and $J_0(u) = J_0(-u)$. It is thus also independent from θ (except θ_0).

In the following, when dealing with classical twisted Yangian, we will choose as generating system the one given in property 2.

3.4 Representations of twisted Yangian

Irreducible representations of twisted Yangians have been studied in [17]. We recall here some of the obtained results[†]. In the following we use the notation

$$n = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad \bar{n} = \left\lceil \frac{N+1}{2} \right\rceil \quad (3.29)$$

Remark that $n = \bar{n}$ iff N is even, and that $N = n + \bar{n}$ in all cases.

3.4.1 Classification

Definition 2 *A representation V of $Y^\pm(N)$ is called lowest weight if there exists a vector $\xi \in V$ such that*

$$\begin{aligned} S^{ij}(u)\xi &= 0 \quad \forall 1 \leq j < i \leq N \\ S^{i,i}(u)\xi &= \mu^i(u)\xi \quad \forall 1 \leq i \leq \bar{n} \end{aligned}$$

ξ is the lowest weight vector of V and $\mu(u) = (\mu^1(u), \dots, \mu^{\bar{n}}(u))$ its lowest weight.

Property 3 *Any finite dimensional irreducible representation of $Y^\pm(N)$ is lowest weight. The lowest weight vector of such a representation is unique (up to multiplication by a scalar). One can chose the basis of $Y^\pm(N)$ in such a way that we have*

$$\mu^i(u) = \prod_{k=1}^{d_i} (1 - \lambda_{(k)}^i u^{-1}) \quad \text{for } Y^-(2n) \quad (3.30)$$

$$(1 + \frac{1}{2}u^{-1})\mu^i(u) = \prod_{k=1}^{2d_i+1} (1 - \lambda_{(k)}^i u^{-1}) \quad \text{for } Y^+(2n) \quad (3.31)$$

$$\mu^i(u) = \prod_{k=1}^{d_i} (1 - \lambda_{(k)}^i u^{-1}) \quad \text{for } Y^+(2n+1) \quad (3.32)$$

where $d_i \in \mathbb{N}$ and $\lambda_{(k)}^i \in \mathbb{C}$.

Property 4 *There is a one-to-one correspondence between finite-dimensional representations of $Y^\pm(N)$ and the families $\{P_1(u), \dots, P_{\bar{n}}(u), \rho(u), \epsilon\}$ where $P_i(u)$ ($\forall i$) are monic polynomials in u with $P_n(u) = P_n(1-u)$, $\rho(u)$ is a formal series in u^{-1} which encodes the values of the Casimir operators of $Y^\pm(N)$ and the parameter ϵ can take the following values:*

[†]Note the difference of convention: $i, j = 1, \dots, N$ (used here) with respect to $-n \leq i, j \leq n$ used in [17]. The correspondence is given by $i \rightarrow \bar{n} - i$ for $i < 0$, and $i \rightarrow n + 1 - i$ for $i \geq 0$. Hence, the apparition of $i = 0$ in [17] is associated to $n \neq \bar{n}$ (which both occur only when $N = 2n + 1$). The transformation $i \rightarrow -i$ used in [17] is translated into $i \rightarrow N + 1 - i$ used in the present article.

- $\epsilon = 1$ for $Y^-(2n)$ (i.e. ϵ is not relevant in this case).
- $\epsilon \in \mathbb{C}$ for $Y^+(2)$, with $P_1(-\epsilon) \neq 0$
- $\epsilon = 1, 2, 3$ or 4 for $Y^+(2n)$, $n > 1$, with the restriction that the values $\epsilon = 1$ and $\epsilon = 3$ have to be identified when $P_n(\frac{1}{2}) \neq 0$.
- $\epsilon = 1$ or 2 for $Y^+(2n+1)$.

By monic polynomials we mean a polynomial of the form $P(u) = \prod_{k=1}^m (u - \gamma_k)$ for some complexes γ_k and some integer m . The monic polynomials are related to the lowest weight of the representation by the relations

$$\frac{\mu^{i+1}(u)}{\mu^i(u)} = \frac{P_i(u+1)}{P_i(u)}, \quad \forall i = 1, \dots, n-2 \quad (3.33)$$

Together with two supplementary relations which depend of the studied twisted Yangian.

In the case of $Y^-(2n)$, the conditions take the form

$$\frac{\mu^n(-u)}{\mu^n(u)} = \frac{P_n(u+1)}{P_n(u)} \quad \text{and} \quad \frac{\mu^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \quad (3.34)$$

The case $Y^+(2)$ is special, the Lie subalgebra being the Abelian $o(2)$. The lowest weight is reconstructed using:

$$\frac{\mu(u)}{\mu(-u)} = \frac{(2u+1)(u+\epsilon)}{(2u-1)(u-\epsilon)} \frac{P(u+1)}{P(u)} \quad (3.35)$$

When considering $Y^+(2n)$, $n > 1$, one has to choose one of the four following possibilities, the choices being labeled by the parameter ϵ :

$$\begin{aligned} \epsilon = 1 & : \quad \frac{\mu^n(-u)}{\mu^n(u)} = \frac{P_n(u+1)}{P_n(u)} \quad \text{and} \quad \frac{\mu^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \\ \epsilon = 2 & : \quad \frac{2u-1}{2u+1} \frac{\mu^n(-u)}{\mu^n(u)} = \frac{P_n(u+1)}{P_n(u)} \quad \text{and} \quad \frac{\mu^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \\ \epsilon = 3 & : \quad \frac{\mu_{\#}^n(-u)}{\mu_{\#}^n(u)} = \frac{P_n(u+1)}{P_n(u)} \quad \text{and} \quad \frac{\mu_{\#}^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \\ \epsilon = 4 & : \quad \frac{2u-1}{2u+1} \frac{\mu_{\#}^n(-u)}{\mu_{\#}^n(u)} = \frac{P_n(u+1)}{P_n(u)} \quad \text{and} \quad \frac{\mu_{\#}^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \end{aligned}$$

where $\mu_{\#}^n(u)$ is deduced from $\mu^n(u)$ by the equations

$$(1 + \frac{1}{2}u^{-1}) \mu^n(u) = \prod_{k=1}^{2d+1} (1 - \lambda^{(k)}u^{-1}) \quad (3.36)$$

$$(1 + \frac{1}{2}u^{-1}) \mu_{\#}^n(u) = (1 + (\lambda^{(2d+1)} + 1)u^{-1}) \prod_{k=1}^{2d} (1 - \lambda^{(k)}u^{-1}) \quad (3.37)$$

Finally, for $Y^+(2n+1)$, one of the two following choices has to be selected, corresponding to the two possible values of ϵ :

$$\begin{aligned} \epsilon = 1 & : \quad \frac{\mu^{\bar{n}}(u)}{\mu^n(u)} = \frac{P_{\bar{n}}(u+1)}{P_{\bar{n}}(u)} \quad \text{and} \quad \frac{\mu^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \\ \epsilon = 2 & : \quad \frac{2u}{2u+1} \frac{\mu^{\bar{n}}(u)}{\mu^n(u)} = \frac{P_{\bar{n}}(u+1)}{P_{\bar{n}}(u)} \quad \text{and} \quad \frac{\mu^n(u)}{\mu^{n-1}(u)} = \frac{P_n(u+1)}{P_n(u)} \end{aligned}$$

3.4.2 Construction from $Y(N)$ representations

All the irreducible finite-dimensional representations of the twisted Yangians can be constructed starting from $Y(N)$ representations. Here, we summarize the construction and refer to [17] for the complete presentation. The basic idea is to try to build a $Y^\pm(N)$ representation as the restriction of an $Y(N)$ one (and eventually doing the subquotient to get an irreducible representation). This provides irreducible representations of $Y^\pm(N)$, but to get a complete classification, one has (sometimes) to add an $o(N)$ representation, understood as an evaluation representation of $Y^+(N)$.

Let $\mu(u)$ be a lowest weight entering in the classification of section 3.4.1, and $V(\mu)$ the corresponding representation. Here, we give (without calculations, see [17] for more details) the lowest weight $\lambda(u)$ and the irreducible representation $L(\lambda)$ of $Y(N)$ which lead to $V(\mu)$ in term of the polynomials $P_i(u)$ of property 4.

We start with the case of $Y(2n)^-$. The lowest weight λ is given by:

$$\begin{aligned} u^s \lambda_i(u) &= \prod_{k=1}^i P_k(u) \prod_{k=i+1}^n P_k(u+1), \quad i = 1, \dots, n \\ u^s \lambda_i(u) &= \prod_{k=1}^n P_k(u+1), \quad i = n+1, \dots, 2n \end{aligned} \tag{3.38}$$

and the corresponding representation $L(\lambda)$ is built as the (subquotient of) the tensor product of s evaluation representations of $Y(N)$, where $s = dg(\lambda_i) = \sum_{i=1}^n dg(P_i)$ and dg is the degree in u for P_i and in u^{-1} for λ .

In the case of $Y(N)^+$, $N \neq 2$: for $\epsilon = 1$, the construction of λ is similar to $Y(2n)^-$. For $\epsilon = 3$ ($N = 2n$), the construction is still the same, but one has to apply the automorphism:

$$S^{ij}(u) \rightarrow S^{i'j'}(u) \quad \text{with} \quad \begin{cases} i' = N+1-i & \text{when } i = n, n+1 \\ i' = i & \text{otherwise} \end{cases} \tag{3.39}$$

For $\epsilon = 2$, one still considers $L(\lambda)$ the lowest weight representation of $Y(N)$ of lowest weight $\lambda(u)$ but tensors it with V_0 the irreducible representation of $o(N)$ with lowest weight $(-\frac{1}{2}, \dots, -\frac{1}{2})$. V_0 can be seen as an evaluation representation of the twisted Yangian, where

the twisted Yangian is embedded into $\mathcal{U}(o(N))$ using the algebra homomorphism[‡]:

$$\begin{aligned} Y^+(N) &\rightarrow \mathcal{U}(o(N)) \\ S(u) &\rightarrow \mathbb{I} + \frac{1}{(u + \frac{1}{2})} \mathbb{F} \end{aligned} \quad (3.40)$$

where $\mathbb{F} = \sum_{i,j} O^{ij} F_{ij}$, with O^{ij} the $o(N)$ -generators and $F_{ij} = E_{ij} + E_{N+1-j, N+1-i}$.

Finally, the case $\epsilon = 4$ is similar to $\epsilon = 3$, but with the use of the automorphism (3.39).

In all the above representations, it may appear that the tensor product is not irreducible: in that case, one has to select the irreducible part of the lowest weight submodule.

For $Y(2)^+$, one constructs the lowest weight representation has the tensor product of s evaluation representations of $Y(N)$ (where s is the degree of $P(u)$), and a $o(2)$ -representation $V(\epsilon)$ of weight $\epsilon - \frac{1}{2}$ (understood as an evaluation representation of $Y(2)^+$, as in (3.40)).

Remark: Contrarily to the Yangian case, in (3.40), the all modes of $S(u)$ have non-zero representation. Indeed, while the commutation relation (3.19) is invariant under the changes $S(u) \rightarrow g(u)S(u)$, the symmetry relation (3.20) restricts $g(u)$ to be even. Thus, the factor $u + \frac{1}{2}$ cannot be removed in (3.40).

4 Folded \mathcal{W} -algebras revisited

It is well-known that the $gl(N)$ Lie algebra can be folded (using an outer automorphism) into orthogonal and symplectic algebras. In the same way, folded \mathcal{W} -algebras have been defined[§] in [12], and shown to be \mathcal{W} -algebras based on orthogonal and symplectic algebras.

We present here a different proof of this property, adapted to our purpose, and generalized to the case of the automorphisms presented in section 3. Indeed, we will see that for $N = 2n$ one can obtain directly the \mathcal{W} -algebras based on $so(2n)$ (as well as those based $sp(2n)$), although there is only one outer automorphism on $gl(2n)$. The situation is exactly the same as the one encountered in the folding of $gl(2n)$: although the folding of this latter algebra (using the automorphism of its Dynkin diagram) leads to $sp(2n)$, it is well-known that the $so(2n)$ algebra can be constructed as the skew-symmetric matrices of $gl(2n)$.

4.1 Automorphism of $gl(Np)$ and $\mathcal{W}_p(N)$

As for the Yangian, one introduces an automorphism of $gl(Np)$ defined by

$$\tau(J_{jm}^{ab}) = (-1)^{j+1} \theta^a \theta^b J_{jm}^{N+1-b, N+1-a} \quad (4.1)$$

[‡]Note that the same type of homomorphism exists also in the case of $Y^-(2n)$ but is not used for the classification of its irreducible finite-dimensional representations.

[§]Strictly speaking, it is the folding of "affine" \mathcal{W} -algebras that has been defined in [12], but the folding of finite \mathcal{W} -algebras can be defined by the same procedure.

where θ^a is defined as in section 3. As for the twisted Yangian, there are only two different cases to be considered: $\theta_0 = 1$, or $\theta_0 = \pm 1$ and $N = 2n$.

To prove that τ is an automorphism of $gl(Np)$, we need the following property of the Clebsch-Gordan coefficients

Property 5 *The Clebsch-Gordan like coefficients obey the rule:*

$$\langle j, m; p, q | r, s \rangle = (-1)^{j+p+r} \langle p, q; j, m | r, s \rangle \quad (4.2)$$

Proof: We first prove the property for $j = m$ and $r = s$. Due to the property of Clebsch-Gordan coefficient, we have $q = r - j$ and we will assume that $q \geq 0$, the proof being similar for $q \leq 0$. One then computes

$$\begin{aligned} \langle j, j; p, q | r, r \rangle &= \frac{(-)^r}{\eta_r} \sum_{k, \ell, m} a_{jj}^k a_{pq}^\ell a_{r, -r}^m \text{tr}(E_{k, k+j} E_{\ell, \ell+q} E_{m+r, m}) \\ &= \frac{(-)^r}{\eta_r} a_{r, -r}^0 \delta_{j+q, r} \sum_{k=1}^{N-j-q} a_{jj}^k a_{pq}^{k+j} \\ \langle p, q; j, j | r, r \rangle &= \frac{(-)^r}{\eta_r} a_{r, -r}^0 \delta_{j+q, r} \sum_{k=1}^{N-j-q} a_{jj}^{k+q} a_{pq}^k \\ &= \frac{(-)^r}{\eta_r} a_{r, -r}^0 \delta_{j+q, r} \sum_{k=1}^{N-j-q} (-1)^{2j+p+q} a_{jj}^{N+1-k-j-q} a_{pq}^{N+1-q-k} \\ &= \frac{(-)^r}{\eta_r} a_{r, -r}^0 \delta_{j+q, r} (-1)^{p+q} \sum_{k=1}^{N-j-q} a_{jj}^k a_{pq}^{k+j} \\ &= (-1)^{j+p+r} \langle j, j; p, q | r, r \rangle \end{aligned}$$

which proves the property for $\langle j, j; p, q | r, r \rangle$. Now, using

$$\begin{aligned} ad_-(M_{jm}^{ab}) &\equiv [e_-, M_{jm}^{ab}] = M_{j, m-1}^{ab} \Rightarrow M_{jm}^{ab} = ad_-^{j-m}(M_{jj}^{ab}) \\ ad_+(M_{jm}^{ab}) &\equiv [e_+, M_{jm}^{ab}] = \frac{j(j+1) - m(m+1)}{2} M_{j, m+1}^{ab} \Rightarrow M_{jm}^{ab} = A_{jm} ad_+^{j+m}(M_{jj}^{ab}) \\ A_{jm} &= \prod_{i=-j}^{m-1} \frac{j(j+1) - i(i+1)}{2} = \frac{(2j)! (j+m)!}{2^{j+m} (j-m)!} \end{aligned}$$

one obtains the two relations

$$\begin{aligned} \langle j, m; p, q | r, r \rangle &= (-1)^{j+m} \langle j, j; p, q+m-j | r, r \rangle \\ \langle j, m; p, q | r, s \rangle &= A_{r, -s} \sum_{i=0}^{r-s} \binom{r-s}{i} B_{j, m}^i B_{pq}^{r-s-i} \langle j, m+i; p, q+r-s-i | r, r \rangle \end{aligned}$$

where

$$B_{j,m}^i = \prod_{k=1}^i \frac{j(j+1) - (m+k)(m+k+1)}{2} = 2^{-i} \frac{(j-m-1)!(j+m+i+1)!}{(j-m-i-1)!(j+m+1)!} \quad (4.3)$$

These two relations ensure that the property 5 is valid for all $\langle j, m; p, q | r, s \rangle$ coefficients. \blacksquare

With this property, it is a simple matter of calculation to show that τ defined in (4.1) is an automorphism of $gl(Np)$.

4.2 Folding $gl(Np)$ and $\mathcal{W}_p(N)$

4.2.1 $gl(Np)$

One considers the subalgebra $\text{Ker}(id - \tau)$ in $gl(Np)$. It is generated by the combinations:

$$K_{jm}^{ab} = J_{jm}^{ab} + \tau(J_{jm}^{ab}) \quad (4.4)$$

which obey the commutation relations:

$$\{K_{jm}^{ab}, K_{k\ell}^{cd}\} = \sum_{r=|j-k|}^{j+k} \sum_{s=-r}^r \langle j, m; k, \ell | r, s \rangle \left(\delta^{bc} K_{r,s}^{ad} - \delta^{ad} (-1)^{j+k+r} K_{r,s}^{cb} + \right. \quad (4.5)$$

$$\left. + (-1)^{j+r} \theta^c \theta^d \delta_{N+1}^{a+c} K_{r,s}^{N+1-d,b} - \theta^a \theta^b \delta_{N+1}^{b+d} (-1)^{k+r} K_{r,s}^{a,N+1-c} \right) \quad (4.6)$$

After a rescaling of K_{jm}^{ab} similar to the one given in theorem 1, one recognizes the algebra $so[(2n+1)p]$ (resp. $sp(2np)$; resp. $so(2np)$) when $N = 2n+1$ (resp. $N = 2n$, $\theta_0 = -1$; resp. $N = 2n$, $\theta_0 = 1$).

Looking at the decomposition of the fundamental of $gl(Np)$ with respect to the principal embedding of $sl(2)$ in $N.sl(p)$ (see [19] and [12] for the technic used here) one shows that the subalgebra $N.sl(p)$, generated by the J_{jm}^{aa} 's, is folded into a $n.sl(p)$ (resp. $n.sl(p) \oplus so(k)$, where $p = 2k+1$ is chosen odd to get N odd) when $N = 2np$ (resp. $N = (2n+1)p$).

In the following, we will denote this subalgebra $[N.sl(p)]^\tau$.

4.2.2 $\mathcal{W}_p(N)$

The situation is more delicate, because we are now dealing with the enveloping algebra of $gl(Np)$, that we denote $\mathcal{U}[gl(Np)] \equiv \mathcal{U}(Np)$. One introduces the coset:

$$\begin{aligned} \mathcal{U}(Np)^\tau &\equiv \mathcal{U}(Np)/\mathcal{K} \quad \text{where } \mathcal{K} = \mathcal{U}(Np) \cdot \mathcal{L} \quad ; \mathcal{L} \text{ generated by } J_{jm}^{ab} - \tau(J_{jm}^{ab}), \forall a, b, j, m \\ \mathcal{W}_p(N)^\tau &\equiv \mathcal{W}_p(N)/\mathcal{J} \quad \text{where } \mathcal{J} = \mathcal{W}_p(N) \cdot \mathcal{I} \quad ; \mathcal{I} \text{ generated by } W_j^{ab} - \tau(W_j^{ab}), \forall a, b, j \end{aligned}$$

We have the property

Property 6 τ is an automorphism of $\mathcal{U}(Np)$ provided with the Dirac brackets:

$$\tau(\{J_{jm}^{ab}, J_{kl}^{cd}\}_*) = \{\tau(J_{jm}^{ab}), \tau(J_{kl}^{cd})\}_* \quad (4.7)$$

Hence, τ is also an automorphism of $\mathcal{W}_p(N)$.

Proof: It is obvious that τ is an automorphism of Poisson brackets on $\mathcal{U}(Np)$. Moreover, due to the form of the constraints (2.15), τ acts as a relabeling (up to a sign) of the constraints:

$$\tau(\varphi_\alpha) = \varphi_{\alpha'} \text{ where } \alpha' \equiv \tau(\alpha) \quad (4.8)$$

and also

$$\tau(\mathcal{C}_{\alpha\beta}) = \mathcal{C}_{\alpha'\beta'} \Rightarrow \tau(\mathcal{C}^{\alpha\beta}) = \mathcal{C}^{\alpha'\beta'} \quad (4.9)$$

This shows that this automorphism is compatible with the set of constraints Φ and thus τ is an automorphism of the Dirac brackets. \blacksquare

Corollary 1 The Dirac brackets provide $\mathcal{W}_p(N)^\tau$ with an algebraic structure.

Proof: We define on $\mathcal{W}_p(N)^\tau$ a bracket which is just the previous Dirac bracket restricted to this coset. Since $\mathcal{W}_p(N)^\tau$ is generated by generators of the form $W + \tau(W)$, we have:

$$\begin{aligned} \{W + \tau(W), W' + \tau(W')\}_* &= \{W, W'\}_* + \{\tau(W), \tau(W')\}_* + \{\tau(W), W'\}_* + \{W, \tau(W')\}_* \\ &= \{W, W'\}_* + \{\tau(W), W'\}_* + \tau(\{W, W'\}_* + \{\tau(W), W'\}_*) \end{aligned}$$

\blacksquare

Indeed we have:

Property 7 The τ -folded algebra $\mathcal{W}_p(N)^\tau$ is the $\mathcal{W}[so[(2n+1)p], n.sl(p) \oplus so(k)]$ algebra (resp. $\mathcal{W}[sp(2np), n.sl(p)]$; resp. $\mathcal{W}[so(2np), n.sl(p)]$ ones) when $N = 2n+1$ and $p = 2k+1$ (resp. $N = 2n$, $\theta_0 = -1$; resp. $N = 2n$, $\theta_0 = 1$).

Proof: On the coset, we have $J_{jm}^{ab} \equiv \tau(J_{jm}^{ab}) \equiv 2K_{jm}^{ab}$. We introduce on $\mathcal{U}(Np)$

$$2D\varphi_\alpha = \varphi_\alpha - \varphi_{\alpha'} = \varphi_\alpha - \tau(\varphi_\alpha) ; 2S\varphi_\alpha = \varphi_\alpha + \varphi_{\alpha'} = \varphi_\alpha + \tau(\varphi_\alpha) \quad (4.10)$$

Since these generators satisfy $D\varphi_\alpha = -D\varphi_{\alpha'}$ and $S\varphi_\alpha = S\varphi_{\alpha'}$ and are in $gl(Np)$, we have

$$\{S\varphi_\alpha, D\varphi_\beta\} \in \mathcal{I} \text{ i.e. } \{S\varphi_\alpha, D\varphi_\beta\} = 0 \text{ on } \mathcal{W}_p(N)^\tau \quad (4.11)$$

Similarly we define

$$D\mathcal{C}_{\alpha\beta} = \{D\varphi_\alpha, D\varphi_\beta\} ; S\mathcal{C}_{\alpha\beta} = \{S\varphi_\alpha, S\varphi_\beta\} ; \quad (4.12)$$

which obey the properties:

$$DC_{\alpha\beta} = DC_{\alpha'\beta'} = -DC_{\alpha'\beta} = -DC_{\alpha\beta'} \quad (4.13)$$

$$SC_{\alpha\beta} = SC_{\alpha'\beta'} = SC_{\alpha'\beta} = SC_{\alpha\beta'} \quad (4.14)$$

$$\mathcal{C}_{\alpha\beta} = SC_{\alpha\beta} + DC_{\alpha\beta} \quad \text{on } \mathcal{W}_p^\tau(N) \quad (4.15)$$

We say that a matrix is τ -antisymmetric when it obeys a relation (4.13), and τ -symmetric when it satisfies (4.14). τ -antisymmetric matrices are orthogonal to τ -symmetric ones:

$$DC \cdot SC = 0 \quad \text{since } (DC \cdot SC)_{\alpha\beta} = \sum_{\gamma} DC_{\alpha\gamma} SC_{\gamma\beta} = \sum_{\gamma'} DC_{\alpha\gamma'} SC_{\gamma'\beta} = - \sum_{\gamma} DC_{\alpha\gamma} SC_{\gamma\beta}$$

$SC_{\alpha\beta}$ is the matrix of constraints of $gl(Np)^\tau$ reduced with respect to $[N.sl(p)]^\tau$. Thus, it is invertible and the associated Dirac brackets define the algebra $\mathcal{W}(gl(Np)^\tau, [N.sl(p)]^\tau)$. It remains to show that, on $\mathcal{W}_p(N)^\tau$, the previously defined Dirac brackets coincide with these latter Dirac brackets.

For that purpose, we use the form $\mathcal{C} = \mathcal{C}_0(\mathbb{I} + \widehat{\mathcal{C}})$, given in [2], where \mathcal{C}_0 is an invertible τ -symmetric matrix and $\widehat{\mathcal{C}}$ is nilpotent (of finite order r). Introducing the τ -symmetrized and antisymmetrized part of $\widehat{\mathcal{C}}$, one deduces

$$\mathcal{C}^{-1} = \mathcal{C}_0^{-1} \sum_{n=0}^r (-1)^n (\widehat{S\mathcal{C}} + D\widehat{\mathcal{C}})^n = \mathcal{C}_0^{-1} \sum_{n=0}^r (-1)^n \left((\widehat{S\mathcal{C}})^n + (D\widehat{\mathcal{C}})^n \right) = SC^{-1} + DC^{-1} \quad (4.16)$$

which shows that DC is also invertible.

On $\mathcal{W}_p(N)^\tau$, we have

$$\{K_{(m)}^{ab}, K_{(n)}^{cd}\}_* = \{K_{(m)}^{ab}, K_{(n)}^{cd}\} - \{K_{(m)}^{ab}, D\varphi_\alpha + S\varphi_\alpha\} \mathcal{C}^{\alpha\beta} \{D\varphi_\beta + S\varphi_\beta, K_{(n)}^{cd}\} \quad (4.17)$$

$$= \{K_{(m)}^{ab}, K_{(n)}^{cd}\} - \{K_{(m)}^{ab}, S\varphi_\alpha\} \mathcal{C}^{\alpha\beta} \{S\varphi_\beta, K_{(n)}^{cd}\} \quad (4.18)$$

$$= \{K_{(m)}^{ab}, K_{(n)}^{cd}\} - \{K_{(m)}^{ab}, S\varphi_\alpha\} (SC^{\alpha\beta} + DC^{\alpha\beta}) \{S\varphi_\beta, K_{(n)}^{cd}\} \quad (4.19)$$

From the τ -antisymmetry of DC^{-1} , we get

$$\{., S\varphi_\alpha\} DC^{\alpha\beta} \{S\varphi_\beta, .\} = \{., S\varphi_{\alpha'}\} DC^{\alpha'\beta} \{S\varphi_\beta, .\} = -\{., S\varphi_\alpha\} DC^{\alpha\beta} \{S\varphi_\beta, .\} = 0 \quad (4.20)$$

which leads to the Dirac brackets:

$$\{K_{(m)}^{ab}, K_{(n)}^{cd}\}_* = \{K_{(m)}^{ab}, K_{(n)}^{cd}\} - \{K_{(m)}^{ab}, S\varphi_\alpha\} SC^{\alpha\beta} \{S\varphi_\beta, K_{(n)}^{cd}\} \quad (4.21)$$

These Dirac brackets are the $\mathcal{W}(gl(Np)^\tau, [N.sl(p)]^\tau)$ algebra ones, by definition of SC . \blacksquare

5 Folded \mathcal{W} -algebras as truncated twisted Yangians

We consider the $\mathcal{W}_p(N)$ algebra in the Yangian basis. The Poisson brackets are

$$\{T_{(q)1}, T_{(r)2}\} = \sum_{s=0}^{\min(p,q,r)-1} (P_{12}T_{(s)1}T_{(q+r-s)2} - T_{(s)2}T_{(q+r-s)1}P_{12}) \quad (5.1)$$

with the convention $T_{(r)} = 0$ for $r > p$. The action of the automorphism τ , both for twisted Yangian and folded $\mathcal{W}_p(N)$ algebra, reads

$$\tau(T_{(m)}) = (-1)^m T_{(m)}^t \quad (5.2)$$

However, from the twisted Yangian point of view, one selects the generators

$$S_{(m)} = \sum_{r+s=m} (-1)^s T_{(r)} T_{(s)}^t$$

while in the folded \mathcal{W} -algebra case, one constrains the generators to $T_{(m)} = (-1)^m T_{(m)}^t$. Although the procedures are different (and indeed lead to different generators), we have:

Theorem 2 *As an algebra, the folded \mathcal{W} -algebra $\mathcal{W}(gl(Np)^\tau, [N.sl(p)]^\tau)$ is isomorphic to the truncation (at level p) of the (classical) twisted Yangian $Y(N)^\tau$.*

More precisely, we have the correspondences:

$$\begin{aligned} Y_p(2n)^- &\longleftrightarrow \mathcal{W}[sp(2np), n.sl(p)] \\ Y_p(2n)^+ &\longleftrightarrow \mathcal{W}[so(2np), n.sl(p)] \\ Y_p(2n+1)^+ &\longleftrightarrow \mathcal{W}[so((2n+1)p), n.sl(p) \oplus so(k)] ; \quad p = 2k+1 \end{aligned} \quad (5.3)$$

Proof: We prove this theorem by showing that the Dirac brackets of the folded \mathcal{W} -algebra coincide with the Poisson brackets

$$\{S_1(u), S_2(v)\} = [r_{12}(u-v), S_1(u)S_2(v)] + S_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)S_2(v) \quad (5.4)$$

with the truncation $S_{(m)} = 0$ for $m > p$.

We start with the $\mathcal{W}_p(N)$ algebra in the truncated Yangian basis and define

$$2\varphi_{(s)} = T_{(s)} - (-1)^s T_{(s)}^t \quad \text{and} \quad 2K_{(s)} = T_{(s)} + (-1)^s T_{(s)}^t \quad (5.5)$$

The folding (of the \mathcal{W} -algebra) corresponds to

$$\varphi_{(s)} = 0 \text{ i.e. } K_{(s)} = (-1)^s K_{(s)}^t \quad (5.6)$$

It is a simple matter of calculation using (5.1), to compute

$$\begin{aligned} 2\{K_{(q)1}, K_{(r)2}\} &= \sum_{s=0}^M \left[P_{12}K_{(s)1}K_{(r+q-s-1)2} - K_{(r+q-s-1)2}K_{(s)1}P_{12} + \right. \\ &\quad \left. + (-1)^{q+s} (K_{(s)1}Q_{12}K_{(r+q-s-1)2} - K_{(r+q-s-1)2}Q_{12}K_{(s)1}) \right] \end{aligned} \quad (5.7)$$

which is equivalent to the relation (5.4) for $S(u) \equiv K(\frac{u}{2})$. The constraint (5.6) is then rewritten as $S^t(-u) = S(u)$. Thus, the folded \mathcal{W} -algebra and the truncated twisted Yangian are defined by the same relations. \blacksquare

5.1 Quantization of \mathcal{W} -algebras

Now that folded \mathcal{W} -algebras have proved to be truncation of twisted Yangians, there quantization is very simple. It takes the form

$$R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v) \quad (5.8)$$

$$\text{with } \begin{cases} R_{12}(x) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} P_{12} ; R'_{12}(x) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} Q_{12} \\ S(u) = \sum_{m=0}^p u^{-m} S_{(m)} ; S_{(0)} = \mathbb{I} \end{cases} \quad (5.9)$$

Let us remark that, contrarily to the quantization of $\mathcal{W}_p(N)$, the quantization of $\mathcal{W}_p(N)^\tau$ indeed modify the commutation relations, adding a non-trivial non-central new term (see equation (3.25) with respect to (3.8), its quantization).

5.2 Center and finite-dimensional irreducible representations

Starting from the classification of finite dimensional representations of the twisted Yangian, one can deduce the ones of the folded \mathcal{W} -algebras.

Property 8 *The finite dimensional irreducible representations of $\mathcal{W}_p(N)^\tau \neq \mathcal{W}_p^+(2)$ are given by the property 4 with the restriction that the polynomials $P_i(u)$ and the parameter ϵ must obey to the following constraints*

$$\sum_{i=1}^{\bar{n}} dg(P_i) \leq \frac{p}{2} \quad \text{and } \epsilon \text{ odd} \quad (5.10)$$

where $dg(P_i)$ is the degree of $P_i(u)$.

In the special case $\mathcal{W}_p^+(2) \equiv \mathcal{W}[so(2p), sl(p)]$, one imposes $dg(P) \leq \frac{p}{2}$ and $\epsilon = \frac{1}{2}$.

Proof: The calculation is the same as for $\mathcal{W}_p(N)$, starting from $Y(N)$ (see [2] for more detail). It essentially relies on the fact that in the tensor product k evaluation representations (of the Yangian $Y(N)$), we have

$$\pi_k(T_{(m)}) = 0 \Leftrightarrow m > k$$

where π_k is the representation morphism. This property is also valid when considering the subquotient of the tensor product. In the case of $Y(N)$, this number corresponds to the

sum of the degrees of the polynomials P_i , hence the condition $k \leq p$ to get a representation of $Y_p(N)$. In the case of $Y^\pm(N)$, considering tensor products of $Y(N)$ -evaluation representations, and since $S(u)$ is quadratic in $T(u)$, one gets:

$$\pi_k(S_{(m)}) = 0 \Leftrightarrow m > 2k$$

Hence, the sum of the degree of the polynomials P_i is (up to the V_0 representation) half of the number of $Y(N)$ -evaluation representations used to build the $Y^\pm(N)$ representation (see constructions in proofs of theorem 5.8, and followings in [17]). When an $o(N)$ -representation V_0 (or a $o(2)$ -representation $V(\epsilon)$ for $Y^+(2)$) is involved, all the generators have non-vanishing representation, and thus cannot be set to zero (see remark at the end of section 3): we have not a representation of the truncated Yangian. Hence, only the values $\epsilon = 1, 3$ are allowed for $Y^\pm(N) \neq Y^+(2)$, and only the value $\epsilon = \frac{1}{2}$ (*i.e.* $V(\epsilon)$ trivial representation) for $Y^+(2)$.

Conversely, starting with a finite dimensional irreducible representation of the truncated twisted Yangian, one can construct a representation of the whole twisted Yangian by representing the remaining generators by zero. This representation is obviously irreducible (since it is for the truncated twisted Yangian), and thus falls into the classification of theorem 4. ■

Again, we can follow the same steps as in [2] to conclude:

Property 9 *The center of $\mathcal{W}_p(N)^\tau$ has dimension $\bar{n}p$ and is generated by the $\bar{n}p$ first even Casimir operators of the underlying $gl(Np)$ algebra.*

Proof: From the Hamiltonian reduction on $[gl(Np)]^\tau$, we know that the center of $\mathcal{W}_p(N)^\tau$ contains the Casimir operators of $[gl(Np)]^\tau$ and that the $\bar{n}p$ first ones are algebraically independent. The property (3.23) and the truncation show that they are the only ones. ■

6 Conclusion

We have shown that the truncation of twisted Yangians $Y^\pm(N)$ are isomorphic to finite \mathcal{W} -algebras based on orthogonal or symplectic algebras. As for $Y(N)$ and finite \mathcal{W} -algebras based on $gl(N)$, this isomorphism allows to classify all the finite dimensional irreducible representations of these \mathcal{W} -algebras, and to determine their center. It provides also a R -matrix formulation of the \mathcal{W} -algebras. However, contrarily to the case of $gl(N)$, the formulation is an $ABCD$ -type one. This confirms the remark already done for $\mathcal{W}_p(N)$ that there seems to be no natural Hopf structure on \mathcal{W} -algebras.

On the other hand, the fact that the isomorphism between Yangians and finite \mathcal{W} -algebras is still valid for $so(m)$ and $sp(m)$ algebras is a good point in favor of the generalization of Yangians. Indeed works are in progress for an R -matrix for all the finite \mathcal{W} -algebras, and a limiting procedure on it should lead to such generalized Yangians.

Finally, note that the supersymmetrization of these construction can also be done [20].

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